EFFICIENT FRONTIER UNDER MARKOWITZ'S SETTING

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We will employ the method of Lagrange multipliers to solve the Markowitz mean-variance problem in its classical quadratic programming formulation:

$$\min w^T \Sigma w$$
$$st. \quad w^T \mu = \mu_g, w^T e = 1$$

where w is the fraction of wealth in each security, μ_g is the mean return vector of the portfolio and Σ is the covariance matrix.

The Lagrangian is defined by

$$L(w,\lambda) = w^T \Sigma w - \lambda_1 \left(w^T \mu - \mu_g\right) - \lambda_2 \left(w^T e - 1\right)$$

hence the optimality conditions are given by

$$\nabla_w L(w,\lambda) = 2\Sigma w - \lambda_1 \mu - \lambda_2 e = 0$$
$$\tfrac{\partial L}{\partial \lambda_1}(w,\lambda) = \mu_g - w^T \mu = 0$$
$$\tfrac{\partial L}{\partial \lambda_2}(w,\lambda) = 1 - w^T e = 0$$

We solve for w in terms of the Lagrange multipliers $w = \frac{1}{2}\lambda_1 \Sigma^{-1}\mu + \frac{1}{2}\lambda_2 \Sigma^{-1} e$. We then substitute this into the other optimality conditions to get a 2-by-2 linear system, i.e.,

$$\begin{aligned}\left(\frac{1}{2}\mu^T \Sigma^{-1}\mu\right)\lambda_1 + \left(\frac{1}{2}\mu^T \Sigma^{-1} e\right)\lambda_2 &= \mu_g \\ \left(\frac{1}{2}\mu^T \Sigma^{-1} e\right)\lambda_1 + \left(\frac{1}{2}e^T \Sigma^{-1} e\right)\lambda_2 &= 1\end{aligned}$$

We further define the following quantities

$$A = e^T \Sigma^{-1} e, B = e^T \Sigma^{-1} \mu, C = \mu^T \Sigma^{-1} \mu, D = AC - B^2$$

we eventually get to

$$w_{optimal} = \frac{1}{D}\left(C\Sigma^{-1} e - B\Sigma^{-1}\mu\right) + \frac{\mu_g}{D}\left(A\Sigma^{-1}\mu - B\Sigma^{-1} e\right)$$

and

$$\sigma_{optimal} = \sqrt{\frac{A\mu_g^2 - 2B\mu_g + C}{D}}$$

The typical efficient frontier will look like this:

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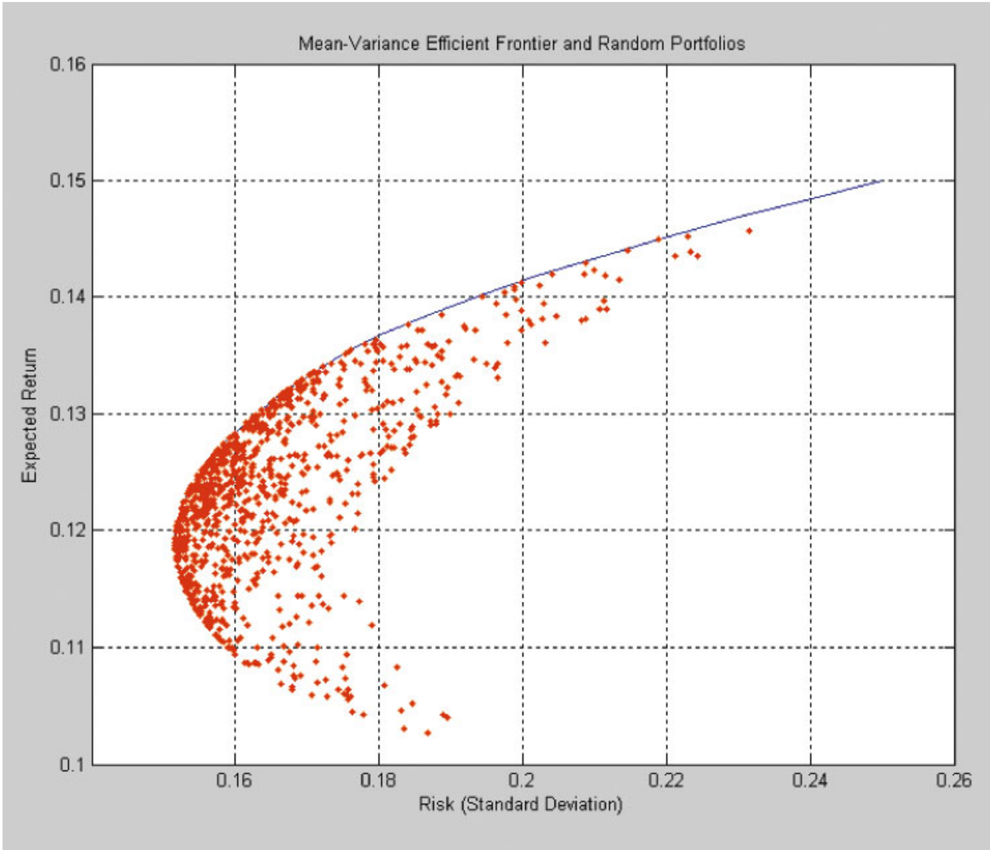

FIGURE 0.1. Efficient Frontier